

CONFORMAL FLAT AK_2 -MANIFOLDS ¹

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In this note we examine AK_2 -manifolds of dimension $2m \geq 6$.

1. Introduction. Let M be a $2m$ - dimensional almost Hermitian manifolds with metric g and almost complex structure J . The Riemannian connection and the curvature tensor are denoted by ∇ and R , respectively. The manifold is said to be a Kähler, or nearly Kähler, or almost Kähler manifold, if

$$(1.1) \quad \begin{aligned} &\nabla J = 0 \quad \text{or} \quad (\nabla_X J)X = 0, \quad \text{or} \\ &g((\nabla_X J))Y, Z) + g((\nabla_Y J))Z, X) + g((\nabla_Z J))X, Y) = 0, \end{aligned}$$

respectively. The corresponding classes of manifolds are denoted by K , NK , AK , respectively. It is well known, that for these classes

$$(1.2) \quad (\nabla_X J)Y + (\nabla_{JX} J)JY = 0$$

holds [2].

For a given class L of almost Hermitian manifolds its subclass L_i is defined by the identity (i), where

- 1) $R(X, Y, Z, U) = R(JX, JY, Z, U)$,
- 2) $R(X, Y, Z, U) = R(JX, JY, Z, U) + R(JX, Y, JZ, U) + R(JX, Y, Z, JU)$,
- 3) $R(X, Y, Z, U) = R(JX, JY, JZ, JU)$.

Then we have $L_1 \subset L_2 \subset L_3$ and $NK = NK_2$, $K = NK_1 = AK_1$, $K = NK \cap AK$ [2].

The Weil conformal curvature tensor C for M is defined by

$$\begin{aligned} C(X, Y, Z, U) &= R(X, Y, Z, U) - \frac{1}{2m-2} \{g(X, U)S(Y, Z) \\ &\quad - g(X, Z)S(Y, U) + g(Y, Z)S(X, U) - g(Y, U)S(X, Z)\} \\ &\quad + \frac{\tau}{(2m-1)(2m-2)} \{g(X, U)g(Y, Z) - g(X, Z)g(Y, U)\}, \end{aligned}$$

where S and τ are the Ricci tensor and the scalar curvature, respectively. It is well known, that (if $m \geq 2$) M is conformal flat if and only if $C = 0$.

Conformal Kähler and nearly Kähler manifolds are classified in [4] and [5]. Here, we shall prove the following theorem:

T h e o r e m. *Let $M \in AK_2$ be a $2m$ -dimensional conformal flat manifold, $m > 2$. Then it is one of the following:*

- a) a flat Kähler manifold;
- b) a 6-dimensional almost Kähler manifold of constant negative sectional curvature;
- c) locally $M_1 \times M_2$, where M_1 (resp. M_2) is a 4-dimensional almost Kähler manifold of constant sectional curvature $-c$ (resp. a 2-dimensional Kähler manifold of constant sectional curvature c), $c > 0$;

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d) locally $M_3 \times M_2$, where M_3 is a 6-dimensional almost Kähler manifold of constant negative sectional curvature $-c$.

R e m a r k 1. We don't know whether there exists an almost Kähler manifold of constant negative sectional curvature of dimension 4 or 6.

R e m a r k 2. If a conformal flat almost Hermitian manifold M satisfies the identity 3), then $S(X, Y) = S(JX, JY)$ and M satisfies also the identity 2).

2. Preliminaries. Let Q be a tensor of type (1.1). According to the Ricci identity,

$$(2.1) \quad (\nabla_X(\nabla_Y Q))Z - (\nabla_Y(\nabla_X Q))Z = R(X, Y)QZ - QR(X, Y)Z .$$

From the second Bianchi identity it follows

$$(2.2) \quad \sum_{i=1}^{2m} (\nabla_{E_i} R)(X, Y, Z, E_i) = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) ,$$

$$(2.3) \quad \sum_{i=1}^{2m} (\nabla_{E_i} S)(X, E_i) = \frac{1}{2} X(\tau) ,$$

where $\{E_i, i = 1, \dots, 2m\}$ is a local orthonormal frame field. We shall assume that $E_{m+i} = JE_i, i = 1, \dots, m$.

Let the tensor S' be defined by

$$S'(X, Y) = \sum_{i=1}^{2m} R(X, E_i, JE_i, JY) .$$

For an AK_2 -manifold the following identities [1,2] hold:

$$(2.4) \quad 2(\nabla_X(S - S'))(Y, Z) = (S - S')((\nabla_X J)Y, JZ) + (S - S')(JY, (\nabla_X J)Z) ,$$

$$(2.5) \quad \sum_{i=1}^{2m} (\nabla_{E_i}(\nabla_{E_i} J))Y = \sum_{i=1}^{2m} J(\nabla_{E_i} J)(\nabla_{E_i} J)Y ,$$

$$(2.6) \quad R(X, Y, Z, U) - R(X, Y, JZ, JU) = \frac{1}{2} g(K(X, Y), K(Z, U)) ,$$

where $K(X, Y) = (\nabla_X J)Y - (\nabla_Y J)X$.

A 2-dimensional almost Hermitian manifold is a Kähler manifold. It follows easily from (2.6), that if M is an almost Kähler manifold of constant curvature c and if $\dim M \geq 4$, then $c \leq 0$ and $c = 0$ if and only if M is a Kähler manifold. On the other hand, an almost Kähler manifold of constant sectional curvature and dimension $2m \geq 8$ is automatically a Kähler manifold [3].

3. Proof of the theorem. From $C = 0$, (2.2) and (2.3) it follows

$$(3.1) \quad (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = \frac{1}{2(2m-1)} \{g(Y, Z)X(\tau) - g(X, Z)Y(\tau)\} .$$

Since $C = 0$ we have

$$S' = \frac{1}{m-1} S - \frac{\tau}{2(m-1)(2m-1)} g .$$

Hence, using (2.4), we find

$$(3.2) \quad 2(\nabla_X S)(Y, Z) = S((\nabla_X J)Y, JZ) + S(JY, (\nabla_X J)Z) - \frac{X(\tau)}{(m-1)(2m-1)}g(Y, Z) .$$

Let $X \perp Y$, JY . According to (3.2) and (1.2),

$$(\nabla_X S)(Y, Y) + (\nabla_X S)(JY, JY) - (\nabla_Y S)(X, Y) - (\nabla_{JY} S)(X, JY) = -\frac{X(\tau)g(Y, Y)}{(m-1)(2m-1)} .$$

The last equality and (3.1) give $X(\tau) = 0$. From $X(\tau) = 0$, (3.1) and (3.2) we obtain

$$(3.3) \quad (\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z) ,$$

$$(3.4) \quad 2(\nabla_X S)(Y, Z) = S((\nabla_X J)Y, JZ) + S(JY, (\nabla_X J)Z) .$$

Now let $p \in M$ and $\{e_i; i = 1, \dots, 2m\}$ be an orthonormal basis of $T_p(M)$, such that $e_{i+m} = Je_i$ and $Se_i = \lambda e_i$ for $i = 1, \dots, m$. Let $\{E_i; i = 1, \dots, 2m\}$ be a local orthonormal frame field, such that $E_i|_p = e_i$ for $i = 1, \dots, 2m$. We have

$$\begin{aligned} & \sum_{i=1}^{2m} (\nabla_{e_i} (\nabla_{e_i} S))(e_j, e_j) = \\ &= \sum_{i=1}^{2m} \{(\nabla_{E_i} (\nabla_{E_i} S))(E_j, E_j) - (\nabla_{\nabla_{E_i} E_i} S)(E_j, E_j) - 2(\nabla_{E_i} S)(\nabla_{E_i} E_j, E_j)\}_p \quad \text{using (3.4)} \\ &= \sum_{i=1}^{2m} \{(\nabla_{E_i} S)((\nabla_{E_i} J)E_j, JE_j) + S((\nabla_{E_i} (\nabla_{E_i} J)E_j, JE_j) \\ & \quad + S((\nabla_{E_i} J)E_j, (\nabla_{E_i} J)E_j)\}_p \quad \text{using (2.5) and (3.4)} \\ &= -\sum_{i=1}^{2m} (\nabla_{e_i} S)((\nabla_{e_i} J)e_j, Je_j) \quad \text{using (3.3)} \\ &= \sum_{i=1}^{2m} (\nabla_{(\nabla_{e_i} J)e_j} S)(e_i, Je_j), \quad \text{using (3.4)} \\ &= \frac{1}{2} \sum_{i=1}^{2m} (\lambda_j - \lambda_i) g((\nabla_{(\nabla_{e_i} J)e_j} J)e_i, e_j) \end{aligned}$$

and using (1.1), we obtain

$$\begin{aligned} (3.5) \quad & \sum_{i=1}^{2m} (\nabla_{e_i} (\nabla_{e_i} S))(e_j, e_j) \\ &= \frac{1}{2} \sum_{i=1}^{2m} (\lambda_j - \lambda_i) \{g((\nabla_{e_i} J)e_j, (\nabla_{e_j} J)e_i) - g((\nabla_{e_i} J)e_j, (\nabla_{e_i} J)e_j)\}. \end{aligned}$$

Because of $X(\tau) = 0$ and (3.3) we have

$$\sum_{i=1}^{2m} (\nabla_{E_j} (\nabla_{E_i} S))(E_i, E_j) = 0 .$$

Using (3.3), we obtain also

$$\sum_{i=1}^{2m} (\nabla_{E_i} (\nabla_{E_i} S))(E_j, E_j) = \sum_{i=1}^{2m} (\nabla_{E_i} (\nabla_{E_j} S))(E_i, E_j) .$$

From the last two equalities and (2.1) it follows

$$(3.6) \quad \sum_{i=1}^{2m} (\nabla_{e_i} (\nabla_{e_i} S))(e_j, e_j) = \sum_{i=1}^{2m} (\lambda_j - \lambda_i) R(e_i, e_j, e_j, e_i) .$$

Now we compute

$$\begin{aligned} & (\nabla_{e_i} (\nabla_{e_j} S))(e_i, e_j) - (\nabla_{e_j} (\nabla_{e_i} S))(e_i, e_j) \quad \text{using (3.4)} \\ &= \frac{1}{2} \{ (\nabla_{e_i} S)((\nabla_{e_j} J)e_i, Je_j) + (\nabla_{e_i} S)(Je_i, (\nabla_{e_j} J)e_j) \\ & \quad + S((\nabla_{e_i} (\nabla_{e_j} J))e_i, Je_j) + S(Je_i, (\nabla_{e_i} (\nabla_{e_j} J))e_j) \\ & \quad - (\nabla_{e_j} S)((\nabla_{e_i} J)e_i, Je_j) - (\nabla_{e_j} S)(Je_i, (\nabla_{e_i} J)e_j) \\ & \quad - S((\nabla_{e_j} (\nabla_{e_i} J))e_i, Je_j) - S(Je_i, (\nabla_{e_j} (\nabla_{e_i} J))e_j) \} \quad \text{using (2.1)} \\ &= \frac{1}{2} \{ (\nabla_{e_i} S)((\nabla_{e_j} J)e_i, Je_j) + (\nabla_{e_i} S)(Je_i, (\nabla_{e_j} J)e_j) \\ & \quad - (\nabla_{e_j} S)((\nabla_{e_i} J)e_i, Je_j) - (\nabla_{e_j} S)(Je_i, (\nabla_{e_i} J)e_j) + (\lambda_j - \lambda_i) R(e_i, e_j, e_j, e_i) \} \end{aligned}$$

and using (3.4) we obtain

$$\begin{aligned} & (\nabla_{e_i} (\nabla_{e_j} S))(e_i, e_j) - (\nabla_{e_j} (\nabla_{e_i} S))(e_i, e_j) = \frac{1}{2} (\lambda_j - \lambda_i) R(e_i, e_j, e_j, e_i) \\ & \quad + \frac{1}{4} (\lambda_j - \lambda_i) \{ g((\nabla_{e_i} J)e_j, (\nabla_{e_j} J)e_i) - g((\nabla_{e_i} J)e_i, (\nabla_{e_j} J)e_j) \}. \end{aligned}$$

On the other hand, (2.1) implies

$$(\nabla_{e_i} (\nabla_{e_j} S))(e_i, e_j) - (\nabla_{e_j} (\nabla_{e_i} S))(e_i, e_j) = (\lambda_j - \lambda_i) R(e_i, e_j, e_j, e_i)$$

and hence we find

$$(3.7) \quad \begin{aligned} & (\lambda_j - \lambda_i) R(e_i, e_j, e_j, e_i) \\ &= \frac{1}{2} (\lambda_j - \lambda_i) \{ g((\nabla_{e_i} J)e_j, (\nabla_{e_j} J)e_i) - g((\nabla_{e_i} J)e_i, (\nabla_{e_j} J)e_j) \} \end{aligned}$$

for all $i, j = 1, \dots, 2m$. If $e_i \neq e_j$, Je_j we have $R(e_i, Je_j, Je_j, e_i) = R(e_i, e_j, e_j, e_i)$ because of $C = 0$. Consequently from (3.7) and (1.2) we derive

$$(3.8) \quad (\lambda_j - \lambda_i) R(e_i, e_j, e_j, e_i) = \frac{1}{2} (\lambda_j - \lambda_i) g((\nabla_{e_i} J)e_j, (\nabla_{e_j} J)e_i)$$

and this is true also for $e_i = e_j$ or $e_i = Je_j$.

From (3.5), (3.6) and (3.8) we obtain

$$(3.9) \quad \sum_{i=1}^{2m} (\lambda_j - \lambda_i) g((\nabla_{e_i} J)e_j, (\nabla_{e_j} J)e_i) = 0$$

for any $j = 1, \dots, 2m$. Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m$. Using (3.9) we find

$$(3.10) \quad \begin{aligned} & \lambda_i = \lambda_1 \quad \text{or} \quad (\nabla_{e_i} J)e_1 = 0 \\ \text{and} \quad & \lambda_i = \lambda_m \quad \text{or} \quad (\nabla_{e_i} J)e_m = 0 \end{aligned}$$

for each $i = 1, \dots, m$. If there exists j , such that $\lambda_1 < \lambda_j < \lambda_m$, then from (3.8) and (3.10) we derive

$$R(e_1, e_j, e_j, e_1) = 0, \quad R(e_m, e_j, e_j, e_m) = 0$$

and because of $C = 0$ this implies $\lambda_1 = \lambda_m$, which is a contradiction. Consequently we have the following two cases:

- 1) $\lambda_i = \lambda_j$ for all $i, j = 1, \dots, m$;
- 2) $\lambda_i = \lambda$ for $i = 1, \dots, n$, $\lambda_i = \mu$ for $i = n + 1, \dots, m$, $\lambda \neq \mu$, $1 \leq n < m$.

In both cases using (3.4) and (3.10), we obtain $\nabla S = 0$ in p . Consequently the Ricci tensor is parallel.

If M is irreducible, it is an Einsteinian manifold and since M is conformal flat, it is of constant sectional curvature. Then the Theorem follows from the results in the end of Preliminaries.

Let M be a reducible non Einsteinian manifold. Then we have the case 2) for each $p \in M$. Now M is locally a product $M_1 \times M_2$, where M_1 and M_2 are almost Kähler manifolds. Let for example $\dim M_1 \geq 4$. Let x, y be orthogonal unit vectors in a point of M_1 and z be a unit vector on M_2 . Because of $C = 0$ we have

$$R(x - z, y + Jz, y - Jz, x + z) = 0$$

or

$$(3.11) \quad R(x, y, y, x) + R(z, Jz, Jz, z) = 0.$$

Hence M_1 is of constant sectional curvature, say $-c$ and consequently $c \geq 0$. If $\dim M_2 = 2$, it follows from (3.11) that M_2 is of constant sectional curvature c . If $\dim M_2 \geq 4$, then M_2 is of constant sectional curvature, say k and from (3.11) we obtain $k = c$. If $c > 0$ this is impossible, because of $\dim M_2 \geq 4$ and if $c = 0$ M is Einsteinian, which is a contradiction.

R E F E R E N C E S

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